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NOTE

Note on the Ruscheweyh Derivatives

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A certain operator $D^{\alpha+p-1}$ defined by convolutions (or Hadamard products) is introduced. The object of this paper is to give an application of the convolution operator $D^{\alpha+p-1}$ to the differential inequalities.

1. INTRODUCTION

Let $A(p)$ denote the class of functions

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \qquad (p \text{ a positive integer}),$$

which are analytic in the unit disc $E = \{z : |z| < 1]$. We denote by $f * g$ the convolution (or Hadamard product) of $f, g \in A(p)$, that is, if

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \in A(p),$$

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \in A(p),$$

then

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \qquad (z \in E).$$

Using the above convolution, we define the operator $D^{\alpha+p-1}$ by

$$D^{\alpha+p-1} f(z) = \frac{z^p}{(1-z)^{\alpha+p}} * f(z),$$

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where $f(z) \in A(p)$ and α is any real number greater than $-p$. We note that [3]

$$D^{n+p-1} f(z) = \frac{z^p \left(z^{n-1} f(z) \right)^{(n+p-1)}}{(n+p-1)!}$$

where n is any integer greater than $-p$.

For our purpose, we introduce

DEFINITION. Let $H(\alpha)$ be the set of complex valued functions $h(r, s, t)$;

$$h(r,s,t) : C^3 \to C \qquad (C \text{ is the complex plane})$$

such that

(i) $h(r,s,t)$ is continuous in a domain in $D \subset C^3$;

(ii) $(1,1,1) \in D$ and $|h(1,1,1)| < J \quad (J > 1)$;

(iii) $$\left| h\left(Je^{i\theta}, \frac{1+m+(\alpha+p)Je^{i\theta}}{\alpha+p+1}, \frac{1}{\alpha+p+2}\left(2+m+(\alpha+p)Je^{i\theta} + \frac{m-m^2+(\alpha+p)mJe^{Ii\theta}+L}{1+m+(\alpha+p)Je^{i\theta}} \right) \right) \right| \geq J,$$

whenever

$$\left(Je^{i\theta}, \frac{1+m+(\alpha+p)Je^{i\theta}}{\alpha+p+1}, \frac{1}{\alpha+p+2}\left(2+m+(\alpha+p)Je^{i\theta} + \frac{m-m^2+(\alpha+p)mJe^{i\theta}+L}{1+m+(\alpha+p)Je^{i\theta}} \right) \right) \in D$$

with $\operatorname{Re} L \geq m(m-1)$ for real θ and for real $m \geq (J-1)/(J+1)$.

2. MAIN RESULT

We begin with the statement of the following lemma due to Miller and Mocanu [1]

LEMMA. *Let $w(z) = a + w_k z^k + \dots$ be regular in E with $w(z) \not\equiv a$ and $k \geq 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, then*

(i) *$z_0 w'(z_0)/w(z_0) = m$ and*

(ii) $\operatorname{Re}[z_0 w''(z_0)/w'(z_0)] \geq m - 1$;

where m is real and

$$m \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Applying the above lemma for the convolution operator $D^{\alpha+p-1}$ we prove

THEOREM. *Let $h(r,s,t) \in H(\alpha)$, and let $f(z)$ belonging to $A(p)$ satisfy*

$$\text{(i)} \quad \left(\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}, \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}, \frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)} \right) \in D \subset C^3 \text{ and}$$

$$\text{(ii)} \quad \left| h\left(\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}, \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}, \frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)} \right) \right| < J$$

for some α, J ($\alpha > -p, J > 1$) and for all $z \in E$. Then we have

$$\left| \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} \right| < J \qquad (z \in E).$$

Proof. We define the function $w(z)$ by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = w(z) \qquad (\alpha > -p)$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z)$ is either analytic or meromorphic in E, $w(0) = 1$, and $w(z) \not\equiv 1$. With the aid of the identity [3].

$$z\left(D^{\alpha+p-1}f(z)\right)' = (\alpha+p)D^{\alpha+p}f(z) - \alpha D^{\alpha+p-1}f(z),$$

we have

$$\begin{aligned} \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} &= \frac{1}{\alpha+p+1}\left[1 + \alpha + \frac{z(D^{\alpha+p}f(z))'}{D^{\alpha+p}f(z)}\right] \\ &= \frac{1}{\alpha+p+1}\left[1 + (\alpha+p)w(z) + \frac{zw'(z)}{w(z)}\right] \end{aligned}$$

and

$$\frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)} = \frac{1}{\alpha+p+2}\Bigg[2+(\alpha+p)w(z)+\frac{zw'(z)}{w(z)}$$

$$+\frac{(\alpha+p)zw'(z)+zw'(z)/w(z)+z^2w''(z)/w(z)-(zw'(z)/w(z))^2}{1+(\alpha+p)w(z)+zw'(z)/w(z)}\Bigg].$$

Suppose that $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z|\leq r_0}|w(z)| = J$. Letting $w(z_0) = Je^{i\theta_0}$ and using the Lemma with $a = 1$ and $k = 1$, we see that

$$\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} = \frac{1}{\alpha+p+1}\left[1+m+(\alpha+p)Je^{i\theta_0}\right]$$

and

$$\frac{D^{\alpha+p+2}f(z_0)}{D^{\alpha+p+1}f(z_0)} = \frac{1}{\alpha+p+2}\Bigg[2+m+(\alpha+p)Je^{i\theta_0}$$

$$+\frac{m-m^2+(\alpha+p)mJe^{i\theta_0}+L}{1+m+(\alpha+p)Je^{i\theta_0}}\Bigg],$$

where $L = z_0^2 w''(z_0)/w(z_0)$ and $m \geq (J-1)/(J+1)$.

Further, an application of (ii) in the Lemma gives

$$\operatorname{Re} L \geq m(m-1).$$

Since $h(r, s, t) \in H(\alpha)$, we have

$$\left|h\left(\frac{D^{\alpha+p}f(z_0)}{D^{\alpha+p-1}f(z_0)}, \frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)}, \frac{D^{\alpha+p+2}f(z_0)}{D^{\alpha+p+1}f(z_0)}\right)\right|$$

$$= \Bigg|h\Bigg(Je^{i\theta_0}, \frac{1+m+(\alpha+p)Je^{1\theta_0}}{\alpha+p+1},$$

$$\frac{1}{\alpha+p+2}(2+m+(\alpha+p)Je^{i\theta_0}$$

$$+\frac{m-m^2+(\alpha+p)mJe^{i\theta_0}+L}{1+m+(\alpha+p)Je^{i\theta_0}}\Bigg)\Bigg)\Bigg| \geq J,$$

which contradicts condition (ii) of the Theorem. Therefore, we conclude that

$$|w(z)| = \left| \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} \right| < J$$

for all $z \in E$. This completes the assertion of the Theorem.

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